

DIRICHLET FORMS AND MARKOV SEMIGROUPS ON C\*-ALGEBRAS<sup>\*)</sup>

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ABSTRACT

We extend the classical theory of Dirichlet forms and associated Markov semigroups to the case of a C\*-algebra with a trace. Semigroups of completely positive maps are characterized by completely positive Dirichlet forms.

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## 1. Introduction

A powerful method for the generation of Markov processes in the commutative case is given by the classical theory of Dirichlet forms and spaces. This theory has its roots in classical potential theory and has been developed particularly since the fundamental work of Beurling and Deny [6]. The theory is closely related with Dynkin's and Hunt's theory of strong Markov processes and has been greatly developed recently in its symmetric  $L_2$ -version particularly by Fukushima and Silverstein, see [20], [21], [30], [31] and [2] - [4].

Since the theory of Dirichlet forms in the commutative case deals with forms which are monotone with respect to a class of contractions applied to certain subalgebras of continuous functions, it is natural to expect a non commutative extension of the theory to the case of  $C^*$ -algebras. It is the purpose of this paper to show that, at least in the case of  $C^*$ -algebras with a trace, this idea can actually be carried through. The outcome are Markov semigroups, i.e. positivity preserving semigroups of maps, and completely Markov semigroups, i.e. semigroup of completely positive maps. Positive and completely positive maps of  $C^*$ -algebras have been the object of many investigations, standard references for foundational work are e.g. [3], [5], [8], [32], [33]. More recently a considerable renewed interest in completely positive maps has arisen particularly in connection with certain foundational problems of non equilibrium statistical mechanics. We allude here to the large body of work on the so called quantum dynamical semigroups and quantum stochastic process, see e.g. [1], [7], [9], [11] - [18], [21] - [27]. Notably a classification of norm continuous completely positive

map on the  $C^*$ -algebra  $\mathcal{B}(\mathcal{H})$  bounded operators on a Hilbert space has been achieved, [27], see also [21]. For other work concerned with the construction and classification of completely positive maps see e.g. [7], [9], [12] - [17], [19], [26].

Markov structures associated with free fermifields are studied in [29] - [35]. In this paper we show in particular that the method of Dirichlet forms permits to obtain a large class of generators of positive and of completely positive maps, which goes beyond the classes obtained previously by other means.

We now summarize shortly the content of the paper.

In Section 2 we introduce the Dirichlet forms on a  $C^*$ -algebra  $A$  with a lower semicontinuous trace  $\tau$ , as positive quadratic forms  $E$  on the hermitian part of  $L^2(A, \tau)$  which have a certain contraction property. We also introduce symmetric Markov semigroups as strong contraction semigroups  $\Phi_t(\cdot)$  on  $L^2(A, \tau)$ , symmetric with respect to the scalar product given by  $\tau$  and such that  $0 \leq x \leq 1$  implies  $0 \leq \Phi_t(x) \leq 1$ . We show that the positive quadratic form given by the infinitesimal generator of a symmetric Markov semigroup  $\Phi_t$  on  $L^2(A, \tau)$  is a Dirichlet form  $E$ , and if  $\Phi_t$  leaves  $A$  invariant then  $E$  is regular in a sense corresponding to the classical one. Conversely we show that the symmetric contraction semigroup generated on  $L^2(A, \tau)$  by a Dirichlet form is a Markov semigroup.

In Section 3 we introduce the concept of a completely Markov semigroup, as a Markov semigroup such that  $\Phi_t$  is completely positive for all  $t \geq 0$ . We prove that  $\Phi_t$  is a completely Markov semigroup on  $L^2(A, \tau)$  if and only if there is a weight  $\rho$  on the algebraic tensor product  $A \otimes \bar{A}$ , with the square

interable elements, such that  $\tau(\phi_t(x)y^*) = \rho(x \otimes y)$ , for all  $x \in A$ ,  $y \in \bar{A}$ . We call a sesquilinear form  $E$  on  $L^2(A, \tau)$  a completely Dirichlet form if  $\sum E(x_{ij}, x_{ij})$  is a Dirichlet form on the hermitian part of  $L^2(A \otimes M_n, \tau \otimes \tau_n)$ , where  $M_n$  are the  $n \times n$  matrices and  $\tau_n$  the corresponding trace, for all  $n$ . We show that a semigroup  $\phi_t$  is completely Markov if and only if the corresponding Dirichlet form  $E$  is completely Dirichlet. A criterium for this is that  $E$  be the **monotone** upwards limit of a sequence of positive bounded forms of the form  $w(x^2) + \rho((x \otimes 1) - 1 \otimes x)^2$ , where  $w$  and  $\rho$  are weights.

## 2. Dirichlet forms and Markov semigroups.

Let  $A$  be a  $C^*$ -algebra with a lower semicontinuous faithful trace  $\tau$  (for the definition see e.g. [11], Ch. V, § 6.). Let  $A_\tau = \{x \in A, \tau(x^*x) < \infty\}$ , then  $A_\tau$  is a two sided ideal of  $A$  and we shall assume that  $A_\tau$  is dense in  $A$  (which, together with the lower semicontinuity implies semifiniteness [ ], ).  $A_\tau$  with the sesqui linear form  $\tau(y^*x)$  is a pre-Hilbert space and its completion will be denoted by  $L^2(A, \tau)$ . We then have that  $A_\tau = A \cap L^2(A, \tau)$  is dense in  $A$  as well as in  $L^2(A, \tau)$ . Since  $\tau(y^*x) = \tau(yx^*)$  (where  $-$  means complex conjugation) we see that  $x \rightarrow x^*$  extends by continuity to an anti-isometry of  $L^2(A, \tau)$ . For any  $a \in A$  we define  $x \rightarrow ax$ ,  $x \in A_\tau$ , and since  $\tau(x^*a^*ax) \leq \|a\|^2 \tau(x^*x)$  we see that the mapping  $x \rightarrow ax$  extends by continuity to an element in  $B(L^2)$  (the space of bounded linear operators on  $L^2(A, \tau)$ ). Hence  $\pi(a)x = ax$  gives us a continuous mapping of  $A$  into  $B(L^2)$  and since  $\pi(a)^* = \pi(a)$  we see that  $\pi$  is a  $*$ -representation of  $A$  on  $L^2$ . Since  $\tau$  is faithful and  $A_\tau$  is dense in  $A$  it follows that  $\pi$  is faithful. Hence  $\pi$  is an injection of  $A$  in  $B(L^2)$ , and we may identify  $A$  with its image  $\pi(A)$ . Now any  $x \in A$  defines a densely defined map of  $L^2(A, \tau)$ , namely  $a \rightarrow xa$  with domain  $A_\tau$ .

Since  $\tau(a^*x^*xa) = \tau(xaa^*x^*) \leq \|a\|^2 \tau(xx^*) = \|a\|^2 \tau(x^*x)$ , we see that, for any  $a \in A$ ,  $x \rightarrow xa$  is strongly continuous in  $L^2(A, \tau)$ , hence it extends by continuity to all of  $L^2(A, \tau)$ . We shall also denote this extended map by  $x \rightarrow xa$ . It is a bounded map from  $L^2 \otimes A$  into  $L^2$ . Hence for any fixed element  $x \in L^2$  we have a densely defined map  $a \rightarrow xa$  with domain  $A_\tau$ . We have that  $\tau(b^*xa) = \tau((x^*b)^*a)$  for any  $a$

and  $b$  in  $A_\tau$ . Hence  $a \rightarrow xa$  has a densely defined adjoint  $a \rightarrow x^*a$  and it is therefore closable. We denote its closure by  $\pi(x)$ .  $\pi(x)$  is then an extension of the representation  $\pi$  on  $A_\tau$  to all of  $L^2(A, \tau)$  mapping the elements in  $L^2(A, \tau)$  into closed (possibly unbounded) operators in  $L^2(A, \tau)$ , such that  $\pi(x)^* = \pi(x^*)$  and one may verify that  $\pi$  is linear in the sense that  $\pi(x) + \pi(y) \subseteq \pi(x+y)$  and  $\pi(\lambda x) = \lambda \pi(x)$ . From the fact that  $\pi$  on  $A$  is faithful it follows that  $\pi$  on  $L^2(A, \tau)$  is one-to-one and therefore allows us to identify  $L^2(A, \tau)$  with a subset of closed operators on  $L^2(A, \tau)$ . We have especially that if  $x$  is invariant under  $*$ , i.e.  $x^* = x$ , then  $\pi(x) = \pi(x)^*$ , so that  $\pi(x)$  is self adjoint. Hence if  $x \in L^2(A, \tau)$  we say that  $x$  is self adjoint if  $x^* = x$  and we also say that  $x \geq 0$  iff  $\pi(x) \geq 0$  and  $0 \leq x \leq 1$  iff  $0 \leq \pi(x) \leq 1$  and so on.

A strongly continuous contraction semigroup  $\Phi_t$ ,  $t \in \mathbb{R}^+$ , on the Hilbert space  $L^2(A, \tau)$  is said to be symmetric iff  $\langle \Phi_t(x), y \rangle = \langle x, \Phi_t(y) \rangle$  where  $\langle , \rangle$  is the scalar product in  $L^2$  and it is said to be Markov iff  $0 \leq x \leq 1$  implies that  $0 \leq \Phi_t(x) \leq 1$ . It is said to be conservative if for any  $a \in A^+$  we have  $\tau(\Phi_t(a)) = \tau(a)$ . In general if  $M \in B(L^2)$  such that  $0 \leq x \leq 1$  implies that  $0 \leq M(x) \leq 1$  we say that  $M$  is Markov. If  $\Phi_t$  is a strongly continuous contraction semigroup on  $L^2(A, \tau)$  then the corresponding resolvent is

$$G_u = \int_0^\infty e^{-ut} \Phi_t \, dt, \quad u \geq 0. \quad (2.1)$$

We have that  $G_u$  satisfies the resolvent equation

$$G_u - G_v = (u-v) G_u G_v \quad (2.2)$$

for  $u$  and  $v$  positive. Moreover we see that  $G_u$  is symmetric iff  $\Phi_t$  is symmetric and  $G_u$  is Markov iff  $\Phi_t$  is Markov. The latter follows from

$$\Phi_t(x) = \lim_{n \rightarrow \infty} \left( \frac{t}{n} G_{t/n} \right)^n(x) \quad (2.3)$$

where the limit is taken in the strong  $L^2$ -sense. Let now  $x = x^*$  in  $L_2$  such that  $\pi(x)$  is bounded. Thus  $-\|\pi(x)\| \leq \pi(x) \leq \|\pi(x)\|$  so that if  $\Phi_t$  is Markov we have that  $-\|\pi(x)\| \leq \Phi_t(x) \leq \|\pi(x)\|$ . Hence  $\|\pi(\Phi_t(x))\| \leq \|\pi(x)\|$ . Since  $\pi$  is one-to-one on  $L^2(A, \tau)$  we have that  $\|x\|_\infty = \|\pi(x)\|$  (operator norm of  $\pi(x)$ ) is a, possibly unbounded, norm on  $L^2$ . We have proved that  $\|\Phi_t(x)\|_\infty \leq \|x\|_\infty$  and in the same way we get that  $\|u G_u(x)\|_\infty \leq \|x\|_\infty$ . From this it follows that  $\Phi_t$  is a strong contraction semigroup on  $L^\infty(A, \tau)$  i.e. on the completion of the domain of  $\|\cdot\|_\infty$  in  $L^2(A, \tau)$  with respect to the  $\|\cdot\|_\infty$ -norm. Hence by the theory of strong contraction semigroups we have that (2.3) also holds in the strong  $L^\infty$ -sense. Since on the other hand the  $L^\infty$ -norm restricted to  $A_\tau = A \cap L^2$  coincides with the  $A$ -norm we have that  $A$  is the  $L^\infty$ -closure of  $A_\tau$ . This together with (2.1) and (2.3) gives us that  $\Phi_t$  leaves  $A$  invariant if and only if  $G_u$  leaves  $A$  invariant. We thus have the following lemma

Lemma 2.1

Any Markov semigroup  $\Phi_t$  on  $L^2(A, \tau)$  extends to a strongly continuous semigroup on  $L^\infty(A, \tau)$ . Moreover this extension leaves  $A \subset L^\infty(A, \tau)$  invariant if and only if the corresponding

Markov resolvent leaves  $A$  invariant. ■

Let now  $x$  and  $y$  be self adjoint elements in  $A$ . Then  $\tau(f(x)g(y))$  is positive for  $f$  and  $g$  positive continuous functions on  $R$ , and it follows from the fact that  $\tau$  is a semifinite and lower semicontinuous trace on  $A$  that there is a positive (possibly unbounded) Radon measure  $\mu_{x,y}$  on  $R \times R$  with support contained in  $\text{Spec}(x) \times \text{Spec}(y)$  such that

$$\tau(f(x)g(y)) = \iint f(\alpha)g(\beta) d\mu_{x,y}(\alpha, \beta). \quad (2.4)$$

From this we get that

$$\tau((f(x) - f(y))^2) = \iint (f(\alpha) - f(\beta))^2 d\mu_{x,y}(\alpha, \beta). \quad (2.5)$$

Let now  $\text{Lip}(R, 0)$  be the Banach space of Lipschitz continuous functions of  $R$  into  $R$  which leave zero invariant.  $\text{Lip}(R, 0)$  is a Banach space in the natural norm

$$\|f\|_{\text{Lip}} = \inf\{m; |f(\alpha) - f(\beta)| \leq m|\alpha - \beta|, \forall (\alpha, \beta) \in R^2\}. \quad (2.6)$$

Let  $\|x\|_2$  be the  $L^2(A, \tau)$  norm then we get from (2.5) that, for  $x$  and  $y$  self adjoint in  $A$  and  $f \in \text{Lip}(R, 0)$ ,

$$\|f(x) - f(y)\|_2 \leq \|f\|_{\text{Lip}} \|x - y\|_2. \quad (2.7)$$

But this tells us that the mapping  $x \mapsto f(x)$  is uniformly con-

tinuous in the  $L^2$ -norm and therefore extends to a mapping from  $L^2_h(A, \tau)$  into  $L^2_h(A, \tau)$  such that (2.7) still holds, where  $L^2_h(A, \tau)$  is the real Hilbertspace of Hermitian elements in  $L^2(A, \tau)$  i.e.  $x \in L^2_h(A, \tau)$  iff  $x = x^*$ ,  $x \in L^2(A, \tau)$ .



Lemma 2.2

Let  $f \in \text{Lip}(R, 0)$  then the mapping  $x \rightarrow f(x)$  defined on the Hermitian part  $A_\tau^h$  of  $A_\tau$  is uniformly continuous in the strong  $L^2$ -norm topology and thus extends to  $L_h^2(A, \tau)$  where it satisfies

$$\|f(x) - f(y)\|_2 \leq \|f\|_{\text{Lip}} \|x - y\|_2 . \quad \blacksquare$$

Let now  $M$  be a bounded operator on  $L^2(A, \tau)$  which is symmetric and Markov. Let  $x$  and  $y$  be in  $A_\tau^h$  then  $\tau(f(x) M(g(x)))$  is positive for  $f$  and  $g$  positive continuous functions on  $R$ , which again makes that we can find a positive Radon measure  $\mu_x$  on  $R^2$  with support on  $\text{Spec}(x) \times \text{Spec}(x)$  such that  $\mu_x(\alpha, \beta) = \mu_x(\beta, \alpha)$  and

$$\tau(f(x) M(g(x))) = \iint f(\alpha) g(\beta) d\mu_x(\alpha, \beta) . \quad (2.8)$$

Since also  $1 - M(1)$  is positive, we have again that there is a positive Radon measure  $\nu_x$  on  $R$  with support on  $\text{Spec}(x)$  such that

$$\tau(f(x)(1 - M(1))) = \int f(\alpha) d\nu_x(\alpha) . \quad (2.9)$$

Consider now the quadratic form  $\tau(x(1 - M)x)$ . We then have

$$\begin{aligned} \tau(f(x)(1 - M)f(x)) &= \tau(f(x)^2(1 - M(1))) + \tau(f(x)^2 M(1) - \\ &\quad f(x) M f(x)) . \end{aligned} \quad (2.10)$$

From (2.8) and (2.9) we therefore have

$$\tau(f(x)(1-M)f(x)) = \int f(\alpha)^2 d\nu_x(\alpha) + \frac{1}{2} \iint (f(\alpha) - f(\beta))^2 d\mu_x(\alpha, \beta) . \quad (2.11)$$

But this immediately gives us that  $\tau(x(1-M)x)$  is a positive form on  $A_\tau^h \times A_\tau^h$ , hence on  $L_h^2 \times L_h^2$ , and that for any  $f \in \text{Lip}(R, 0)$  and  $x \in A_\tau^h$  we have

$$\tau(f(x)(1-M)f(x)) \leq \|f\|_{\text{Lip}}^2 \tau(x(1-M)x) , \quad (2.12)$$

and by continuity (2.12) also holds for  $x \in L_h^2(A, \tau)$ .

### Lemma 2.3

Let  $M$  be a bounded operator on  $L_h^2(A, \tau)$  such that  $M$  is symmetric and Markov. Then the form  $\langle x, (1-M)x \rangle$  on  $L_h^2$  is symmetric and positive. Moreover for any  $f \in \text{Lip}(R, 0)$  we have that

$$\langle f(x), (1-M)f(x) \rangle \leq \|f\|_{\text{Lip}}^2 \langle x, (1-M)x \rangle$$

where  $\langle x, x \rangle$  is the square norm in  $L_h^2$ .  $\blacksquare$

Let  $E(x, x)$  be a positive closed quadratic form on  $L_h^2(A, \tau)$  (not necessarily bounded) with dense domain  $D(E)$ . We say that  $E$  is a Dirichlet form if in addition to being densely defined, positive and closed it satisfies the condition that  $D(E)$  is invariant under the mapping  $x \rightarrow f(x)$  for any  $f \in \text{Lip}(R, 0)$  and

$$E(f(x), f(x)) \leq \|f\|_{\text{Lip}}^2 E(x, x) . \quad (2.13)$$

Corollary 2.4

If  $M$  is a bounded operator on  $L^2_h(A, \tau)$  which is symmetric and Markov then  $\langle x, (1-M)x \rangle$  is a regular Dirichlet form.

We say that  $E$  is a regular Dirichlet form if in addition  $A^h_\tau \cap D(E)$  is norm dense in  $A^h$  and is also dense in  $D(E)$  where  $D(E)$  is equipped with its natural norm  $\|x\|_E^2 = E(x, x) + \tau(x^2)$ . ■

Let  $\epsilon(x, x)$  be a positive quadratic form defined on a domain  $D(\epsilon)$  which is a linear subspace of  $A^h$  (the real Banach space of Hermitian elements in  $A$ ). We say that  $\epsilon(x, x)$  is a Markov form on  $A$  if, for any  $\delta > 0$ , there exists a non-decreasing real function  $\varphi_\delta(t), t \in \mathbb{R}$ , satisfying the following conditions

$$\varphi_\delta(t) = t \quad \text{for } 0 \leq t \leq 1 \quad (2.14)$$

$$|\varphi_\delta(t)| \leq t \quad \text{and} \quad -\delta \leq \varphi_\delta(t) \leq 1+\delta \quad \text{for all } t$$

such that if  $x \in D(\epsilon)$  then  $\varphi_\delta(x) \in D(\epsilon)$  and

$$\epsilon(\varphi_\delta(x), \varphi_\delta(x)) \leq \epsilon(x, x). \quad (2.15)$$

(The definition of a Markov form for a commutative  $C^*$ -algebra was given by Fukushima [ ].) We say that a positive quadratic form  $\epsilon(x, x)$  on  $A^h$  is compatible with the trace  $\tau$  if  $D(\epsilon) \cap A^h_\tau$  is dense in  $L^2(A, \tau)$  and the restriction of  $\epsilon$  to  $D(\epsilon) \cap A^h_\tau$  is a closable form in  $L^2_h(A, \tau)$ . If  $\epsilon$  is compatible with  $\tau$  then its closure  $E$  defines a unique non negative self adjoint operator  $H$  on  $L^2_h(A, \tau)$  such that  $D(E)$  is the same as  $D(H^{\frac{1}{2}})$ , the domain of  $H^{\frac{1}{2}}$ , and  $E(x, x) = \langle x, Hx \rangle$ .

$H$  is then of course the infinitesimal generator of a strongly continuous symmetric semigroup on  $L^2_h(A, \tau)$ . We shall return to this point later. We have now the following theorem connecting Markov and Dirichlet forms. The proof of this theorem is the same as in the commutative case where it was given by Fukushima [ ].

Theorem 2.5

Let  $\epsilon$  be a Markov form on a  $C^*$ -algebra  $A$ . If  $\tau$  is a semifinite and lower semicontinuous faithful trace such that  $A_\tau$  is norm dense in  $A$  and  $\epsilon$  is compatible with  $\tau$ , then the closure  $E$  of  $\epsilon$  on  $L^2_h(A, \tau)$  is a Dirichlet form.  $E$  is a regular Dirichlet form if  $D(\epsilon) \cap A^h_\tau$  is norm dense in  $A^h$ . Let now  $\Phi_t$  be a symmetric Markov semigroup and let  $G_u$  be the corresponding resolvent. Let  $H$  be the infinitesimal generator of  $\Phi_t$ . We know that  $H$  is a positive self adjoint operator on  $L^2(A, \tau)$  such that  $\Phi_t = e^{-tH}$  and

$$Hx = \lim_{t \rightarrow 0} \frac{1}{t}(1 - \Phi_t)x \quad (2.16)$$

or

$$Hx = \lim_{u \rightarrow \infty} u(1 - uG_u)x \quad (2.17)$$

in the sense that  $x \in D(H)$  iff any one of the strong limits above exists and in which case  $Hx$  is given by the right hand side of (2.16) or (2.17).

The following lemma is an immediate consequence of the spectral decomposition of strongly continuous symmetric contraction semigroups on Hilbert spaces.

Lemma 2.6

For any  $x \in L^2_h(A, \tau)$  we have that

$$\frac{1}{t} \langle x, (1 - \Phi_t)x \rangle \quad \text{and} \quad u \langle x, (1 - u G_u)x \rangle$$

increase as  $t \downarrow 0$  and  $u \uparrow \infty$ . Each of these expressions remains bounded if and only if  $x \in D(H^{\frac{1}{2}})$ , where  $H$  is the infinitesimal generator of  $\Phi_t$  and in this case we have, with  $E(x, x) = \|H^{\frac{1}{2}}x\|_2^2$ , that

$$E(x, x) = \lim_{t \downarrow 0} \frac{1}{t} \langle x, (1 - \Phi_t)x \rangle = \lim_{u \rightarrow \infty} u \langle x, (1 - G_u)x \rangle. \quad \blacksquare$$

Now by corollary (2.4)  $\frac{1}{t} \langle x, (1 - \Phi_t)x \rangle$  is a regular Dirichlet form so that for any  $f \in \text{Lip}(R, 0)$  we have

$$\frac{1}{t} \langle f(x), (1 - \Phi_t)f(x) \rangle \leq \|f\|_{\text{Lip}}^2 \frac{1}{t} \langle x, (1 - \Phi_t)x \rangle. \quad (2.18)$$

Since by lemma 2.6  $D(E)$  consists exactly of those elements  $x$  for which  $\frac{1}{t} \langle x, (1 - \Phi_t)x \rangle$  remains finite as  $t \downarrow 0$  we get by (2.18) that  $D(E)$  is invariant under  $x \rightarrow f(x)$ . Moreover by taking the limit  $t \downarrow 0$  in (2.18) we get that  $E(f(x), f(x)) \leq E(x, x)$ . Hence we have proved that  $E(x, x) = \|H^{\frac{1}{2}}x\|_2^2$  is a Dirichlet form. Let us now consider the  $L^\infty$ -extension of  $\Phi_t$  and let us assume that  $\Phi_t(A) \subset A$ . This is obviously equivalent with the assumption that our semigroup on  $L^2(A, \tau)$  comes from a semigroup of Markov maps of  $A$  which are symmetric with respect to  $\tau$ . We then have that  $G_u(A) \subset A$  and since

$G_u$  is the resolvent for a contraction semigroup on  $A$  we have by the theory of contraction semigroups (see e.g. ) that  $G_u(A)$  is dense in  $A$ . However  $G_u(A) = D(H) \subset D(H^{\frac{1}{2}}) = D(E)$ . On the other hand for any positive self adjoint operator  $H$  we have that  $D(H)$  is dense in  $D(H^{\frac{1}{2}})$  in the natural norm in  $D(H^{\frac{1}{2}})$ . Hence we have proved in this case that  $E$  is a regular Dirichlet form. We summarize these results in the following theorem.

Theorem 2.7

Let  $E(x,x) = \|H^{\frac{1}{2}}x\|_2^2$  be the positive quadratic form given by a symmetric Markov semigroup  $\phi_t$  on  $L^2(A,\tau)$ , then  $E(x,x)$  is a Dirichlet form. If  $\phi_t(A) \subset A$  then  $E(x,x)$  is a regular Dirichlet form.  $\blacksquare$

Consider now an arbitrary Dirichlet form  $E(x,x)$  and let  $H$  be the corresponding positive self-adjoint operator on  $L_h^2(A,\tau)$  so that  $E(x,x) = \|H^{\frac{1}{2}}x\|_2^2$ . Let  $G_u = (u+H)^{-1}$ ,  $u > 0$  be the corresponding resolvent. Set

$$E_u(x,x) = E(x,x) + u\langle x,x \rangle. \quad (2.19)$$

Then we have for any  $x$  and  $y$  in  $D(E)$  that

$$E_u(G_u y, x) = \langle y, x \rangle \quad (2.20)$$

and

$$E_u(x - uG_u y, x - uG_u y) = E(x,x) + u[\langle x,x \rangle + u\langle y, G_u y \rangle - 2\langle x, y \rangle] = \quad (2.21)$$

$$E(x,x) + u\langle x-y, x-y \rangle - u\langle y, (1-G_u)y \rangle.$$

Thus  $E(x,x) + u\|x-y\|_2^2$  has a unique minimum for  $x = uG_u y$ .

Let now  $y$  be such that  $y = f(y)$  where  $f$  is a contraction of the real line leaving zero fixed, e.e.  $f \in \text{Lip}(\mathbb{R}, 0)$  with  $\|f\|_{\text{Lip}} \leq 1$ . Then by the assumption that  $E(x,x)$  is a Dirichlet form together with lemma 2.2 we get, since  $y = f(y)$ , that

$$\begin{aligned} E(f(x), f(x)) + u\|f(x) - y\|_2^2 &= E(f(x), f(x)) + u\|f(x) - f(y)\|_2^2 \\ &\leq E(x, x) + u\|x - y\|_2^2. \end{aligned} \quad (2.22)$$

If we take now for  $x$  the minimal point  $x = uG_u y$  then we have by (2.22) that  $f(x)$  also gives a minimal point and by uniqueness of this minimal point we get  $f(x) = x$ . Hence we have proved that if  $y = f(y)$  then  $uG_u y = f(uG_u y)$ . Take now  $f(\alpha) = (0 \vee \alpha) \wedge 1$  which is obviously a contraction of the real line leaving zero fixed, then we get that  $uG_u$  is Markov for all  $u$ . Since this implies that  $\Phi_t$  is Markov we have the following

### Theorem 2.8

Let  $E(x,x)$  be a Dirichlet form on  $L^2_h(A, \tau)$ , and let  $H$  be the positive self adjoint operator given by  $E$ . Then the symmetric contraction semigroup generated by  $H$  is a Markov semigroup on  $L^2(A, \tau)$ .

### 3. Completely Markov semigroups.

Let now  $\Phi_t$  be a strongly continuous contraction semigroup on the Hilbert space  $L^2(A, \tau)$ . We said that  $\Phi_t$  is Markov if  $x \in L^2(A, \tau)$  such that  $0 \leq x \leq 1$  implies that  $0 \leq \Phi_t(x) \leq 1$ . Recalling that  $0 \leq x \leq 1$  was defined by considering the closure  $L_x$  of the operator  $a \rightarrow xa$  defined for  $x \in L^2(A, \tau)$  with domain  $A_\tau \subset L^2(A, \tau)$  (the notation  $\pi(x)$  was used for  $L_x$  in the previous section) and then  $0 \leq x \leq 1$  was equivalent to  $0 \leq L_x \leq 1$ .  $L^\infty(A, \tau)$  was then defined as the completion in the norm  $\|x\|_\infty = \|L_x\|$  of the linear subspace of  $L^2(A, \tau)$  consisting of elements  $x$  such that  $L_x$  is a bounded operator on  $L^2(A, \tau)$ , and hence  $L_x$  extends to an isomorphism of  $L^\infty(A, \tau)$  with the weakly closed subalgebra  $L \subset B(L^2(A, \tau))$  obtained by taking the weak closure of the set  $L_x$  with  $x \in A \subset L^\infty(A, \tau)$ . Hence if  $\pi$  is the representation of  $A$  by left translation on  $L^2(A, \tau)$  considered in the previous section we have  $L = \pi(A)''$  (the weak closure of  $\pi(A)$ ). Therefore we see that  $\Phi_t$  is Markov if and only if it extends to  $L^\infty(A, \tau) = L$  and defines a positively preserving semigroup  $\Phi_t$  on the  $W^*$ -algebra  $L$  such that  $\Phi_t(1) \leq 1$ .

Let  $A$  be a  $C^*$ -algebra and  $M_n$  the  $C^*$ -algebra of  $n \times n$  complex matrices. The elements  $X \in A \otimes M_n$  may be represented by  $X = \{x_{ij}\}$ , a  $n \times n$  matrix with elements  $x_{ij} \in A$ , and if  $Y = \{y_{ij}\}$  then  $XY = \{\sum_k x_{ik} y_{kj}\}$ . If  $\Phi$  is a map of  $A$  we define  $\Phi_n$  as the map of  $A \otimes M_n$  given by

$$\Phi_n(X) = \{\Phi(x_{ij})\} \quad \text{for } X = \{x_{ij}\} \quad (3.1)$$



i.e.  $\Phi_n(x) = \Phi \otimes 1_n$ , where  $1_n$  is the identity of  $M_n$ . A linear map  $\Phi$  of  $A$  is said to be completely positive iff  $\Phi_n$  is a positive map of the  $C^*$ -algebra  $A \otimes M_n$  for any  $n$ . Especially we have that a completely positive map is positive. Similarly we say that a map  $\Phi$  is completely Markov iff  $\Phi_n$  is Markov for any  $n$ , and we say that a semigroup  $\Phi_t$  is completely Markov iff  $\Phi_t$  is completely Markov for any  $t \geq 0$ .

Let now, for any  $x \in L^2(A, \tau)$ ,  $R_x$  be the closure of the mapping given by  $a \rightarrow ax$  with domain  $A$ , and let  $R$  be the weak closure of the set of elements in  $B(L^2(A, \tau))$  of the form  $R_x$ . It is easy to see that  $R = \pi(A)'$  i.e.  $R$  is equal to the commutant of  $\pi(A)$ , so that  $R' = L$  and  $L' = R$ . We also remark that, while the restriction of  $L_x$  to  $x \in A$  is a faithful  $*$ -representation of  $A$  on  $L^2(A, \tau)$ , we have that the restriction of  $R_x$  to  $x \in A$  is a faithful anti  $*$ -representation of  $A$  on  $L^2(A, \tau)$ . Let  $S$  be the anti isometry of  $L^2(A, \tau)$  given by  $S_x = x^*$ , then it follows immediately that  $R_x = SL_x^* S = SL_x^* S$  and therefore  $\|R_x\| = \|L_x^*\| = \|L_x\|$ . Hence  $R_x$  extends by continuity to an isometry of  $L^\infty(A, \tau)$  onto  $R \subset B(L^2(A, \tau))$ .

For any  $C^*$ -algebra  $A$  we define the conjugate algebra  $\bar{A}$  which is identical with  $A$  apart from the scalar multiplication which in  $\bar{A}$  is defined by  $(\lambda, a) \rightarrow \bar{\lambda}a$ ,  $\lambda \in \mathbb{C}$  and  $a \in \bar{A}$ , where  $\bar{\lambda}$  is the complex conjugate of  $\lambda$ . Let now  $\Phi \in B(L^2(A, \tau))$  be Markov i.e.  $0 \leq x \leq 1$  implies  $0 \leq \Phi(x) \leq 1$ . If  $\Phi$  is completely Markov then  $\Phi_n = \Phi \otimes 1_n$  is Markov on  $A \otimes M_n$ . Let now  $\tau_n$  be the natural trace on  $M_n$ , then  $\tau \otimes \tau_n$  is a trace on  $A \otimes M_n$  so that  $\tau \otimes \tau_n(XY) \geq 0$  if  $x \geq 0$  and  $Y \geq 0$  in

$A \otimes M_n$  . Hence

$$\begin{aligned} \tau \otimes \tau_n(\Phi_n(X)Y) &= \sum_{ij} \tau(\Phi(x_{ij})y_{ji}) \\ &= \sum_{ij} \tau(\Phi(x_{ij})y_{ij}^*) \geq 0 \end{aligned} \quad (3.2)$$

whenever  $X = \{x_{ij}\}$  and  $Y = \{y_{ij}\}$  are positive elements in  $A \otimes M_n$  . Remark that since  $Y = Y^*$  we have  $y_{ij} = y_{ji}^*$  .

Let now  $x_1, \dots, x_n$  be in  $A_\tau$  then  $u = (\sum_i x_i \otimes y_i)^* \sum_i (x_i \otimes y_i) = \sum_{ij} x_i^* x_j \otimes y_i^* y_j$  is a positive element in  $A \otimes \bar{A}$  (we have considered  $y_1, \dots, y_n$  to be in  $\bar{A}$ ), where  $A \otimes \bar{A}$  is the algebraic tensor product of  $A$  and  $\bar{A}$  . We now define a linear functional  $\rho$  on the algebraic tensor product  $A \otimes \bar{A}$  with domain of definition  $A_\tau \otimes \bar{A}_\tau$  , where  $\bar{A}_\tau$  is the image of  $A_\tau$  in  $\bar{A}$  , by

$$\rho(x \otimes y) = \tau(\Phi(x)y^*) , \quad (3.3)$$

then

$$\rho(u) = \sum_{ij} \rho(x_i^* x_j \otimes y_i y_j^*) = \sum_{ij} \tau(\Phi(x_i^* x_j) y_j y_i^*) . \quad (3.4)$$

Now  $X = \{x_i^* x_j\}$  and  $Y = \{y_i y_j^*\}$  are obviously positive elements in  $A \otimes M_n$  so by (3.2) we have that  $\rho(u) \geq 0$  . Hence  $\rho$  is a positive linear functional on  $A \otimes \bar{A}$  or a weight on  $A \otimes \bar{A}$  with domain  $A_\tau \otimes \bar{A}_\tau$  .

Let now conversely  $\Phi \in B(L^2(A, \tau))$  be Markov and let us assume that  $\tau(\Phi(x)y^*) = \rho(x \otimes y)$  , where  $\rho$  is a weight on the algebraic tensor product  $A \otimes \bar{A}$  with domain  $A_\tau \otimes \bar{A}_\tau$  . The densely defined weight  $\rho$  gives rise to a representation of  $A \otimes \bar{A}$  on a Hilbert space  $K$  by the GNS construction and let  $\eta$  be the corre-

sponding mapping from  $A \otimes \bar{A}$  into  $K$ . Consider now the linear mapping from  $L^2(A, \tau)$  into  $K$  with dense domain  $A_\tau$  given by

$$Vy = \eta(1 \otimes y^*) . \quad (3.5)$$

Since we have

$$\begin{aligned} (Vy, Vz) &= (\eta(1 \otimes y^*), \eta(1 \otimes z^*)) \\ &= \rho((1 \otimes z^*)^*(1 \otimes y^*)) = \rho(1 \otimes zy^*) \\ &= \tau(\Phi(1)yz^*) = \langle \Phi(1)y, z \rangle , \end{aligned} \quad (3.6)$$

where  $( , )$  is the inner product in  $K$  and  $\langle , \rangle$  is the inner product in  $L^2(A, \tau)$ . It follows from (3.6) that  $V^*V = \Phi(1) \leq 1$  so that  $V$  is bounded and extends by continuity to a bounded linear map of  $L^2(A, \tau)$  into  $K$ . We observe that if  $\Phi(1) = 1$  then  $V$  is an isometric imbedding of  $L^2(A, \tau)$  into  $K$ . Let  $\pi_\rho$  be the representation of  $A \otimes \bar{A}$  given by  $\rho$ , by the GNS construction, and set  $\pi(x) = \pi_\rho(x \otimes 1)$ .  $\pi$  is then a  $*$ -representation of  $A$  on  $K$ . Then for  $y$  and  $z$  in  $A_\tau$  we have

$$\begin{aligned} \langle V^*\pi(x)Vy, z \rangle &= (\pi_\rho(x \otimes 1)Vy, Vz) \\ &= (\pi_\rho(x \otimes 1) \eta(1 \otimes y^*), \eta(1 \otimes z^*)) \\ &= (\eta(x \otimes y^*), \eta(1 \otimes z^*)) = \rho((1 \otimes z^*)^*(x \otimes y^*)) \\ &= \rho(x \otimes zy^*) = \tau(\Phi(x)yz^*) = \langle \Phi(x)y, z \rangle . \end{aligned} \quad (3.7)$$

Hence we get

$$\Phi(x) = V^* \pi(x) V \quad (3.8)$$

which is a completely positive map since it is the composition of two completely positive maps, namely  $x \mapsto \pi(x)$  and  $y \mapsto V^* y V$ . We summarize these results in the following theorem

Theorem 3.1

Let  $\Phi$  be a bounded map of  $L^2(A, \tau)$  into  $L^2(A, \tau)$  which is Markov. Then  $\Phi$  is completely Markov if and only if there is a weight  $\rho$  on the algebraic tensor product  $A \otimes \bar{A}$  with domain containing  $A_\tau \otimes \bar{A}_\tau$  such that

$$\tau(\Phi(x)y^*) = \rho(x \otimes y) . \quad \blacksquare$$

Let now  $\Phi_t$  be a strongly continuous one parameter contraction semigroup on  $L^2(A, \tau)$  which is symmetric and completely Markov. By the previous theorem we have a one parametric family of weights  $\rho_t$  on  $A \otimes \bar{A}$  such that  $\frac{1}{2t} \langle \Phi_t(x), y \rangle = \rho_t(x \otimes y)$  and since for  $x \in L^2(A, \tau)$  we have

$$\langle (1 - \Phi_t)x, x \rangle = \langle (1 - \Phi_t(1))x, x \rangle + \langle \Phi(1)x, x \rangle - \langle \Phi_t(x), x \rangle \quad (3.9)$$

then

$$\frac{1}{t} \langle (1 - \Phi_t)x, x \rangle = w_t(x^2) + \rho_t((x \otimes 1 - 1 \otimes x)^2) \quad (3.10)$$

where  $t \cdot w_t(x) = \tau((1 - \Phi_t(1))x)$  is a weight on  $A$ , since  $0 \leq \Phi_t(1) \leq 1$ . By lemma 2.6 we have that  $\frac{1}{t} \langle (1 - \Phi_t)x, x \rangle \uparrow E(x, x)$  as  $t \downarrow 0$ , where  $E(x, x)$  is the Dirichlet form corresponding to  $\Phi_t$ . Observe that  $\rho(x \otimes y) = \overline{\rho(y \otimes x)}$  since  $\Phi_t$  is

symmetric. We say that  $\rho$  is a symmetric weight on  $A \otimes \bar{A}$ . Hence we have that the Dirichlet form  $E(x, x)$  corresponding to a symmetric and completely Markov semigroup is the increasing limit of bounded Dirichlet forms of the type  $E_t(x, x) = w_t(x^2) + \rho_t((x \otimes 1 - 1 \otimes x)^2)$  where  $w$  and  $\rho$  are weights on  $A$  and  $A \otimes \bar{A}$  respectively.

On the other hand assume that  $w_Y$  is a weight on  $A$  and  $\rho_Y$  is a weight on the algebraic tensor product  $A \otimes \bar{A}$ , such that

$$E_Y(x, x) = w_Y(x^2) + \rho_Y((x \otimes 1 - 1 \otimes x)^2) \quad (3.11)$$

is a bounded positive bilinear form on  $L^2_h(A, \tau)$  such that  $0 \leq E_{Y_1}(x, x) \leq E_{Y_2}(x, x)$  for  $Y_1 \leq Y_2$  and let us assume that there is a closed bilinear form  $F$  such that  $E_Y(x, x) \leq F(x, x)$  for any  $Y$  in  $R^+$ . Let now  $x \in A_\tau$  such that  $x = x^*$ . From the spectral representation theorem and the fact that  $w_Y$  and  $\rho_Y$  are weights we get that

$$w_Y(f(x)) = \int f(\alpha) d\nu_Y(\alpha)$$

and (3.12)

$$\rho_Y((f(x) \otimes 1 - 1 \otimes f(x))^2) = \iint (f(\alpha) - f(\beta))^2 d\mu_Y(\alpha, \beta),$$

where  $\nu_Y$  and  $\mu_Y$  are positive Radon measures on  $R$  and  $R \times R$  respectively, depending on  $x$ , and with support on  $\text{Spec}(x)$  and  $\text{Spec}(x) \times \text{Spec}(x)$  respectively. Hence we have that for  $x \in A_\tau \cap L^2_h(A, \tau)$

$$E_Y(f(x), f(x)) = \int f(\alpha)^2 d\nu_Y(\alpha) + \iint (f(\alpha) - f(\beta))^2 d\mu_Y(\alpha, \beta). \quad (3.13)$$

So far  $f \in \text{Lip}(R, 0)$  we have, for  $x \in A_\tau \cap L_h^2(A, \tau)$ , that

$$E_Y(f(x), f(x)) \leq \|f\|_{\text{Lip}}^2 E_Y(x, x), \quad (3.14)$$

which obviously implies that  $E_Y$  with domain  $A_\tau \cap L_h^2(A, \tau)$  is Markov and since  $E_Y$  is bounded it is a Dirichlet form.

If  $E(x, x)$  is a sesqui linear form on  $L^2(A, \tau)$  we set

$$E_n(X, X) = \sum_{ij} E(x_{ij}, x_{ij}) \quad (3.15)$$

for  $X = \{x_{ij}\} \in L_h^2(A \otimes M_n, \tau \otimes \tau_n)$ . Since  $L^2(A \otimes M_n, \tau \otimes \tau_n) = L^2(A, \tau) \otimes L^2(M_n, \tau_n)$  and  $E_n = E \otimes 1_n$  where  $1_n$  is the form given by the identity in  $L^2(M_n, \tau_n)$  we see that  $E_n$  is closable if and only if  $E$  is closable and if  $E$  is closed then  $D(E_n) = D(E) \otimes M_n$ . We say that  $E$  is a completely Markov form if  $E_n$  is a Markov form for each  $n \geq 1$ , and we say that  $E$  is a completely Dirichlet form iff  $E_n$  is a Dirichlet form for each  $n$ . By what is above we see that a Dirichlet form which is completely Markov is completely Dirichlet.

Consider now  $E_Y(x, x)$  given by (3.11). Since  $E_{Y,n}$  is given in the way of (3.11) by  $w_Y \otimes \tau_n$  and  $\rho_Y \otimes (\tau_n \otimes \tau_n)$ , where  $\tau_n$  is the natural trace in  $M_n$ , it follows as above that  $E_{Y,n}(X, X)$  is a Dirichlet form. Therefore  $E_Y(x, x)$  is a completely Dirichlet form.

Let now  $E_Y(x, x)$  be an increasing sequence of bounded Dirichlet forms i.e.  $0 \leq E_{Y_1}(x, x) \leq E_{Y_2}(x, x)$  for  $Y_1 \leq Y_2$  and let us also assume that  $E_Y(x, x) \leq F(x, x)$  where  $F$  is a closed form. It is then well known that  $E(x, x) = \lim_{Y \uparrow \infty} E_Y(x, x)$ , with

domain  $D(E)$  consisting of those  $x$  for which the limit is finite, is a closed form. From (3.14) we have that  $x = x^* \in D(E)$  and  $f \in \text{Lip}(R, 0)$  implies that  $f(x) \in D(E)$  and  $E(f(x), f(x)) \leq \|f\|_{\text{Lip}}^2 E(x, x)$  so that  $E(x, x)$  is a Dirichlet form. If moreover  $E_Y$  is of the form (3.11) then we have that  $E_{Y,n}$  are Dirichlet forms, and since  $E_{Y,n} = E_Y \otimes 1_n \uparrow E \otimes 1_n = E_n$  it follows that  $E_n$  is a Dirichlet form so that  $E_Y$  is a completely Dirichlet form. We summarize these results in the following theorem

Theorem 3.2

Let  $E$  be a Dirichlet form on  $L^2(A, \tau)$  and  $\Phi_t$  the corresponding symmetric Markov semigroup. Then  $\Phi_t$  is completely Markov if and only if  $E$  is completely Dirichlet. Moreover  $E$  is completely Dirichlet if and only if there exists an increasing sequence of positive bounded forms  $E_Y$ ,

$0 \leq E_{Y_1}(x, x) \leq E_{Y_2}(x, x)$  for  $Y_1 \leq Y_2$  and  $E(x, x) = \lim_{Y \uparrow \infty} E_Y(x, x)$  with domain  $D(E)$  equal to the set of  $x$  for which this limit is finite, and  $E_Y$  has the form

$$E_Y(x, x) = w_Y(x^2) + \rho_Y((x \otimes 1 - 1 \otimes x)^2)$$

where  $w_Y$  and  $\rho_Y$  are weights on  $A$  and on the algebraic tensorproduct  $A \otimes A$  respectively, where  $\bar{A}$  is the conjugate  $C^*$ -algebra corresponding to  $A$ .

#### 4. Normal contractions on C\*-algebras.

For the commutative C\*-algebra  $C(X)$ ,  $X$  a locally compact space one says that  $v \in C(X)$  is a normal contraction of  $u \in C(X)$  iff  $|v(\alpha)| \leq |u(\alpha)|$  and  $|v(\alpha) - v(\beta)| \leq |u(\alpha) - u(\beta)|$  for any  $\alpha$  and  $\beta$  in  $X$ . We see that if  $v(\alpha)$  is a normal contraction of  $u(\alpha)$  then  $v(\alpha)$  is continuous in the topology generated by  $u(\alpha)$  hence there is a continuous mapping  $f$  of the real line  $R$  into itself such that  $v(\alpha) = f(u(\alpha))$ , and from the two inequalities  $|v(\alpha)| \leq |u(\alpha)|$  and  $|v(\alpha) - v(\beta)| \leq |u(\alpha) - u(\beta)|$  it follows that  $f(\alpha)$  may be taken as a contraction of  $R$  leaving zero fixed i.e.  $f(0) = 0$  and  $|f(\alpha) - f(\beta)| \leq |\alpha - \beta|$ .

Let now  $C_h(X)$  be the real (self adjoint) part of  $C(X)$ , then  $v(x)$  is a normal contraction of  $u(\alpha)$  iff  $v^2 \leq u^2$  and  $(v(\alpha) - v(\beta))^2 \leq (u(\alpha) - u(\beta))^2$ . Now  $u(\alpha) - u(\beta)$  may be considered as an element in  $C_h(X) \otimes C_h(X) \subseteq C_h(X \times X)$  where the tensorproduct is the algebraic tensor product, namely  $u(\alpha) - u(\beta) \equiv u \otimes 1 - 1 \otimes u \in C_h(x) \otimes C_h(x)$ . Hence we may write the condition for normal contraction as

$$v^2 \leq u^2 \quad \text{and} \quad (v \otimes 1 - 1 \otimes v)^2 \leq (y \otimes 1 - 1 \otimes u)^2 \quad (4.1)$$

where the first inequality is in  $C(X)$  and the second inequality is in  $C(X) \otimes C(X)$ . Let now  $A$  be an arbitrary C\*-algebra with a unite. If  $x$  and  $y$  are in  $A_h$  (the self adjoint part of  $A$ ) then we say that  $x$  is a normal contraction of  $y$  if

$$x^2 \leq y^2 \quad \text{and} \quad (x \otimes 1 - 1 \otimes x)^2 \leq (y \otimes 1 - 1 \otimes y)^2 \quad (4.2)$$

where the first inequality is in  $A$  and the second inequality



is in  $A \otimes A$ , the algebraic tensor product of  $A$  with it self. Since  $x$  and  $y$  are self adjoint  $x = x^*$  and  $y = y^*$  we may also consider the second inequality to be in  $A \otimes \bar{A}$ , where  $\bar{A}$  is the conjugate algebra. If  $y \in A_h$  and  $x = f(y)$  where  $f(0) = 0$  and  $|f(\alpha) - f(\beta)| \leq |\alpha - \beta|$  it follows easily from what is said before that  $x$  is a normal contraction of  $y$  because in this case  $x$  and  $y$  are in the same commutative subalgebra. We shall now see that if  $x$  is a normal contraction of  $y$  than  $x = f(y)$  where  $f(0) = 0$  and  $|f(\alpha) - f(\beta)| \leq |\alpha - \beta|$ .

We may assume that  $A \subseteq B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ , and as we are only interested in the subalgebra of  $A$  generated by  $x$  and  $y$  we may also assume that  $\mathcal{H}$  is separable. Let  $L^2(\mathcal{H})$  be the Hilbert space of Hilbert-Schmidt operators on  $\mathcal{H}$ . Then  $L^2(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}$  and  $B(\mathcal{H}) \otimes B(\mathcal{H})$  is naturally imbedded in  $B(L^2(\mathcal{H}))$  by the correspondance  $a \otimes b \in B(\mathcal{H}) \otimes B(\mathcal{H})$  goes to mapping  $m \rightarrow a m b^*$  contained in  $B(L^2(\mathcal{H}))$ ,  $\overline{B(\mathcal{H})}$  is the conjugate algebra of  $B(\mathcal{H})$ . If we denote  $m \rightarrow am$  by  $L(a)$  and  $m \rightarrow mb$  by  $R(a)$  we have the imbedding of  $B(\mathcal{H}) \otimes B(\mathcal{H})$  into  $B(L^2(\mathcal{H}))$  is given by  $a \otimes b \rightarrow L(a)R(b^*)$ . It is easy to see that this imbedding is a faithful \*-representation. Hence we have that  $(x \otimes 1 - 1 \otimes x)^2 \leq (y \otimes 1 - 1 \otimes y)^2$  if and only if

$$(L(x) - R(x))^2 \leq (L(y) - R(y))^2 \quad (4.3)$$

in  $B(L^2(\mathcal{H}))$ . Hence  $(x \otimes 1 - 1 \otimes x)^2 \leq (y \otimes 1 - 1 \otimes y)^2$  is equivalent with the statement that for any  $m \in L^2(\mathcal{H})$ , i.e. for any  $m \in B(\mathcal{H})$  with  $\text{tr}(m^*m) < \infty$  we have that

$$(m, (L(x) - R(x))^2 m) \leq (m, (L(y) - R(y))^2 m) \quad (4.4)$$

where  $(\cdot, \cdot)$  is the inner product in  $L^2(\mathcal{H})$  so that (4.4) is equivalent with

$$\text{tr}(m^*[x[x, m]]) \leq \text{tr}(m^*[y, [y, m]]) . \quad (4.5)$$

Here  $[x, m] = (L(x) - R(x))m$  i.e. the commutator of  $x$  and  $m$ . From (4.5) we get that

$$\text{tr}([x, m]^*[x, m]) \leq \text{tr}([y, m]^*[y, m]) \quad (4.6)$$

Take now  $m$  to commute with  $y$  then by (4.6) the Hilbert-Smidt norm of  $[x, m]$  is zero so that  $x$  commutes with  $m$ . Hence the commutant of  $x$  contains the commutant of  $y$ , and therefor  $x$  is in the commutative algebra generated by  $y$ . Since  $x$  and  $y$  is in the same commutative subalgebra we have by the argument above that  $x = f(y)$  where  $f$  is a contraction of the real line i.e.  $|f(\alpha) - f(\beta)| \leq |\alpha - \beta|$  for any real  $\alpha$  and  $\beta$ . We summarize these results in the following theorem.

#### Theorem 4.1

Let  $x$  and  $y$  be self adjoint elements in a  $C^*$ -algebra  $A$  with unite. Then

$$(x \otimes 1 - 1 \otimes x)^2 \leq (y \otimes 1 - 1 \otimes y)^2$$

in the algebraic tensor product  $A \otimes A$  if and only if there is a contraction  $f$  of the real line ( $|f(\alpha) - f(\beta)| \leq |\alpha - \beta|$ ) such that  $x = f(y)$ .

Moreover  $x$  is a normal contraction of  $y$  if and only if there is a contraction  $f$  of the real line such that  $f(0) = 0$  and  $x = f(y)$ .

Combining this theorem with the results of section 2 we get the following theorem.

Theorem 4.2

Let  $E(x,x)$  be a closed positive form on  $L^2(A,\tau)$  where  $\tau$  is a lower semicontinuous faithful trace on the  $C^*$ -algebra  $A$ . Then  $E(x,x)$  is a Dirichlet form if and only if for  $y \in D(E)$  and  $x$  a normal contraction of  $y$  then  $x \in D(E)$  and  $E(x,x) \leq E(y,y)$ .

From the proof of theorem 4.1 we have the following lemma

Lemma 4.3

If  $A \subset B(\mathcal{H})$  and  $x$  and  $y$  are in  $A$ . Then  $x$  is a normal contraction of  $y$  if and only if

$$x^2 \leq y^2$$

and for any  $m \in B(\mathcal{H})$  such that  $\text{tr}(m^*m) < \infty$  we have that

$$\text{tr}([x,m]^*[x,m]) \leq \text{tr}([y,m]^*[y,m]) .$$

Let now  $\tau$  be a lower semicontinuous faithful trace on the  $C^*$ -algebra  $A$  and assume that  $A \subset B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ . If  $M$  is a positive selfadjoint operator on  $\mathcal{H}$  (not necessarily bounded) and  $m_i \in B(\mathcal{H})$  with  $\text{tr}(m_i^*m_i) < \infty$  we consider the form on  $L^2(A,\tau)$  given by

$$E(x,x) = \text{tr}(x^2M) + \sum_{i=1}^{\infty} \text{tr}([x,m_i]^*[x,m_i]) . \quad (4.7)$$

Then if  $E(x,x)$  is closable on  $L^2(A,\tau)$  then by theorem 4.2 and lemma 4.3 we have that  $E(x,x)$  is Dirichlet. It follows

easily that it is completely Dirichlet because the form  $E^{(n)}(x,x)$  on  $A \otimes M_n$  is obtained by replacing  $M$  by  $M \otimes 1_n$  and  $m_i$  by  $m_i \otimes 1_n$ , and therefor  $E^{(n)}(x,x)$  is again Dirichlet. Hence we have

Corollary 4.4

If  $A \subset B(\mathcal{H})$  and  $M \geq 0$  is a self adjoint operator (not necessarily bounded) and  $m_i \in B(\mathcal{H})$  satisfy  $\text{tr}(m_i^* m_i) < \infty$  then if

$$E(x,x) = \text{tr}(x^2 M) + \sum_{i=1}^{\infty} \text{tr}([x, m_i]^* [x, m_i])$$

is closable on  $L^2(A, \tau)$  then  $E(x,x)$  is completely Dirichlet.

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